Eddy Currents

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Abstract

This is a quick calculation to see how eddy currents will affect our design. There are at least two phenomena to consider (a) heating (b) B-fields generated by eddy currents that will reduce the generating field. In the geometry I am considering, I am going to assume that everything is thin, i.e. both the wall of the end plate and the wall of the tube that encloses the ferrites are thin when compared to the skin depth. Without this assumption, I can compare the heating and the behaviour of the B-field between the disk and the tube due to eddy currents.

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I. INTRODUCTION

There are two major problems with eddy currents generated by the solenoidal ramp

- Heating by the eddy currents
- Slow down of the ramp because of the B-field generated by the eddy currents, i.e. there is a diffusion time constant that comes from solving the diffusion equation of the magnetic vector potential. See for example Knoepfel [1].

II. THEORY

I am going to partition the problem into two parts:

- Eddy currents that are on the end walls. I will assume that the end wall is a disk.
- Eddy currents that flow on the tube. I will calculate this for an infinitely long thin walled tube. This should be ok because if the wall is thin, there is negligible effect on the B-field that penetrates the tube transversely on this face.

See Fig. 1.

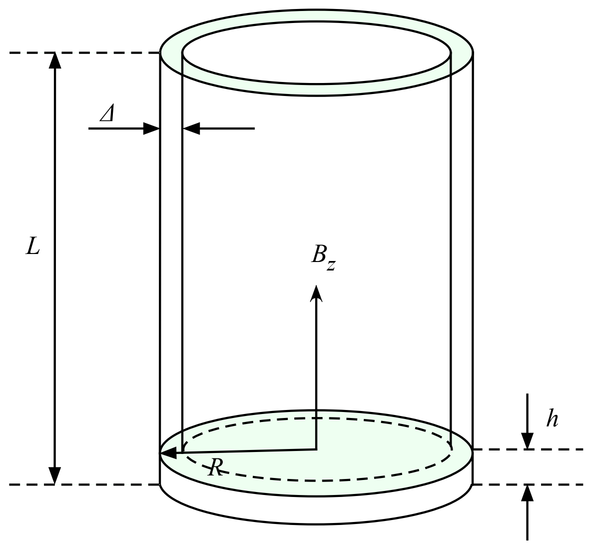

FIG. 1. The long tube of length L with wall thickness Δ has an end plate. The end plate is a disk of radius R and thickness h. The changing magnetic field B_z is normal to the disk face and is parallel to the tube wall.

But first, let me calculate the skin depth for copper so that I can see whether the approximations that I will use later is valid. The skin depth is given by the formula

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} \tag{1}$$

where δ is the skin depth in m, f is the frequency in Hz, μ is the permeability of the material in H/m, and σ is the electrical conductivity in S/m.

For copper, $\mu = 1.256 \times 10^{-6}$ H/m, $\sigma = 5.96 \times 10^{7}$ S/m and $f = 15$ Hz, I get

$$\delta_{\mathrm{Cu}} = 1.7 \text{ cm} \tag{2}$$

A. Eddy currents flowing in a disk

I am going to look at how the effect of eddy currents in the disk shown in Fig. 2 first. The accurate calculation of eddy currents for a disk that has thickness comparable to the skin depth is actually rather involved. See for example [2, 3]. However, I can get a good approximation of the effect if I assume that $h \ll \delta$, i.e. the current density is constant in the disk. I can calculate the current in this case by using the integral form of Faraday's Law. (c.f. Knoepfel (page 211)[1], who gets the same answer as I do by solving a a rather complicated partial differential equation.)

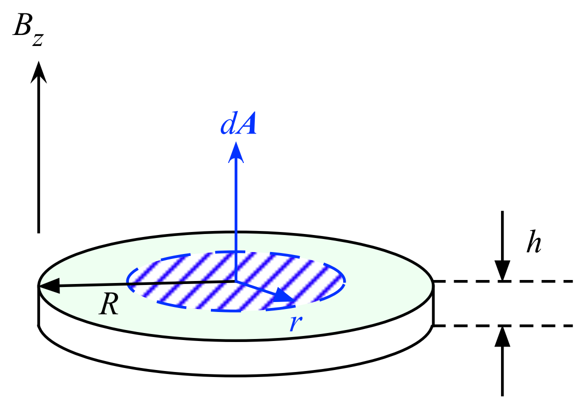

FIG. 2. The end wall of the structure shown in Fig. 1. The integral path is a circle of radius r that encloses the hatched area A.

$$\oint \boldsymbol{E} \cdot d\boldsymbol{\ell} = -\frac{d}{dt}\int_A \boldsymbol{B} \cdot d\boldsymbol{A} \tag{3}$$

In the case that I am considering, the closed path integral of $\boldsymbol{E}$ is around a circle of radius r and the enclosed area is A. The disk area does not change and so the only time varying term on the rhs is contained in $\boldsymbol{B}$. I can perform the integral easily to get

$$\begin{aligned} 2\pi r E_\theta &= -\dot{B}_z \pi r^2 \\ \therefore \quad E_\theta &= -\frac{1}{2}\dot{B}_z r \end{aligned} \tag{4}$$

where E_θ is the azimuthal component of the E-field.

Let me suppose that the disk is made of a material that has conductivity σ, then from Ohm's Law, I have

$$j_\theta = \sigma E_\theta = -\frac{1}{2}\sigma \dot{B}_z r \tag{5}$$

where j_θ is the current density that is flowing in the hatched circle shown in Fig. 2.

Hence the current dI_θ that flows between r and $r + dr$ is

$$dI_\theta = j_\theta \times (h\ dr) = -\frac{1}{2}\sigma \dot{B}_z h r\ dr \tag{6}$$

where I have assumed that the skin depth is much larger than h.

Thus, the total current I_θ that flows from 0 to R is

$$I_\theta = -\frac{1}{2}\sigma \dot{B}_z h \int_0^R r\ dr = -\frac{1}{4}\sigma \dot{B}_z h R^2 \tag{7}$$

The consequence of I_θ is that there is heating of the disk and a B-field that is pointing in the opposite direction to B_z. I will consider these consequences in the next two subsections.

1. *Power loss from heating*

The power loss in the cross sectional area $a = h\,dr$ is

$$dP_{\text{disk}} = \left(\frac{2\pi r}{\sigma a}\right) \times dI_\theta^2 = \left(\frac{2\pi r}{\sigma a}\right) \times \left(\frac{1}{2}\sigma \dot{B}_z a r\right)^2 \tag{8}$$

because the resistance the current sees in the circular path that it travels is $2\pi r/\sigma a$ and, of course, the power loss simply comes from given by $I^2 R$.

When I integrate the above, I get the total power loss P_{disk} due to eddy currents flowing in this disk

$$P_{\text{disk}} = \frac{\pi}{2}\sigma \dot{B}_z^2 h \int_0^R r^3\ dr = \frac{\pi}{8}\sigma \dot{B}_z^2 h R^4 \tag{9}$$

The scaling to the 4th power of R is definitely something we need to take into account in the cavity shell design! Note that Eq. 9 is the instantaneous power loss. I can perform another time integral to get the average power loss over 1 cycle if B_z is periodic. But I'm leaving this as is for now.

2. Magnetic field reduction

I can use the Biot-Savart Law to calculate the field B' that arises to the eddy current I_θ. For simplicity, I will only calculate the B' along the z axis that passes throught the centre of the disk. See Fig. 3. To calculate B'_z, I start with a current loop dI_θ that flows around a loop of radius r that generates the dB'_z field

$$dB'_z = \frac{\mu_0}{4\pi}\left(\int_C \frac{(dI_\theta)\boldsymbol{d\ell}\times\boldsymbol{\rho}}{\rho^3}\right)_z = \frac{\mu_0}{2}\frac{r^2}{(r^2+z^2)^{3/2}}dI_\theta \tag{10}$$

$$= -\frac{\mu_0}{4}\sigma\dot{B}_z h\frac{r^3}{(r^2+z^2)^{3/2}}dr \tag{11}$$

where $\boldsymbol{\rho}$ is the vector from the loop to the point of interest. In this special case, both dB'_θ and dB'_r are zero at the point P shown in Fig. 3 by symmetry.

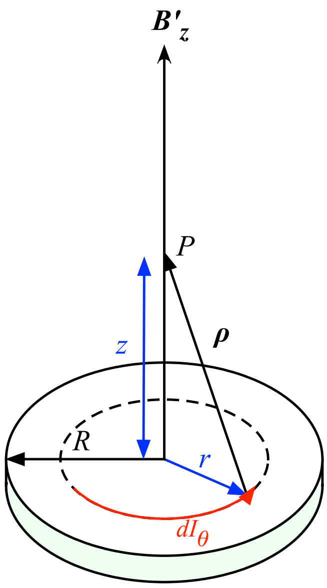

FIG. 3. The B-field is calculated at point P due to the eddy current dI_θ flowing in the disk. $\boldsymbol{\rho}$ is the vector that goes from the current contribution $\boldsymbol{d\ell}$ in the loop to P.

To get the total contribution to B'_z, I have to perform the integral from 0 to R to get

$$
\begin{aligned}
B'_z &= -\frac{\mu_0}{4}\sigma \dot{B}_z h \int_0^R dr\, \frac{r^3}{(r^2+z^2)^{3/2}} \\
&= -\frac{\mu_0}{4}\sigma \dot{B}_z h \frac{\left(z-\sqrt{R^2+z^2}\right)^2}{\sqrt{R^2+z^2}}
\end{aligned}
\tag{12}
$$

Right at the centre of the disk $z = 0$, the eddy currents reduce B_z by

$$
B'_z(z=0) = -\frac{\mu_0}{4}\sigma \dot{B}_z h R \tag{13}
$$

B. Eddy currents flowing in a tube

The calculation of the eddy currents flowing around the tube that has a finite length and thickness is complicated. Rather than slog away with this geometry, I will use the infinitely long thin shell approximation instead which is a lot easier. In this approximation, I am ignoring the transverse edges (i.e. the part that connects to the disk shown in Fig. 1) and $\Delta \ll \delta$ so that the current density is constant within the shell. I think this approximation is ok because for a thin enough wall, the effect on B_z at the transverse edge is negligible.

I can use the same current density equation Eq. 5 as before, but with $r = R$

$$
j_\theta = -\frac{1}{2}\sigma \dot{B}^i_z R \tag{14}
$$

And note that in the thin shell approximation, j_θ is constant in the shell, the current that is flowing is simply given by

$$
dI_\theta = j_\theta \times (\Delta\, dz) = -\frac{1}{2}\sigma \dot{B}^i_z R \Delta\, dz \tag{15}
$$

1. *Power losses from heating*

The power loss in the cross sectional area $a = \Delta\, dz$ is

$$
\begin{aligned}
dP_{\text{tube}} &= \left(\frac{2\pi R}{\sigma a}\right) \times dI_\theta^2 = \left(\frac{2\pi R}{\sigma a}\right) \times \left(\frac{1}{2}\sigma \dot{B}^i_Z R a\right)^2 \\
&= \frac{\pi}{2}\sigma \left(\dot{B}^i_z\right)^2 R^3 \Delta\, dz
\end{aligned}
\tag{16}
$$

Hence, the instantaneous total power loss over length L is

$$
P_{\text{tube}} = \frac{\pi}{2}\sigma \left(\dot{B}^i_z\right)^2 R^3 L \Delta \tag{17}
$$

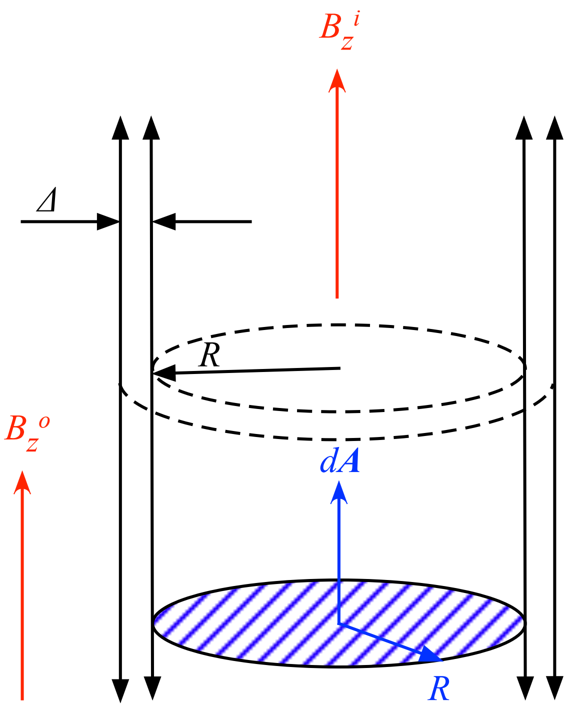

FIG. 4. The tube shown here has been made infinite in length so that there are no transverse faces in the problem. The integral path is a circle of radius R that encloses the hatched area A. δ is the skin depth. Note that I have partitioned B_z into two parts because the wall causes a retardation of the drive field B_z^o.

Notice that the losses is linearly dependent on the thickness of the tube Δ and is only to the third power in R.

2. Magnetic field diffusion time

From Ampere's Law, I can integrate around the closed loop shown in Fig. 4 to get

$$\begin{aligned} & \oint \boldsymbol{B} \cdot d\boldsymbol{\ell} = \mu_0 \int_S \boldsymbol{j} \cdot d\boldsymbol{S} \\ \Rightarrow \quad & B_z^i - B_z^o = \mu_0 j_\theta \times \Delta \end{aligned} \tag{18}$$

because the L's cancel on both sides. I can substitute in j_θ from Eq. 14 into the above to get

$$\begin{aligned} B_z^i - B_z^o &= -\left(\frac{1}{2}\mu_0 \sigma R \Delta\right) \dot{B}_z^i \\ \Rightarrow \quad \dot{B}_z^i + \frac{B_z^i}{\tau} &= \frac{B_z^o}{\tau} \end{aligned} \tag{19}$$

where $\tau = \mu_0 \sigma R \Delta / 2$.

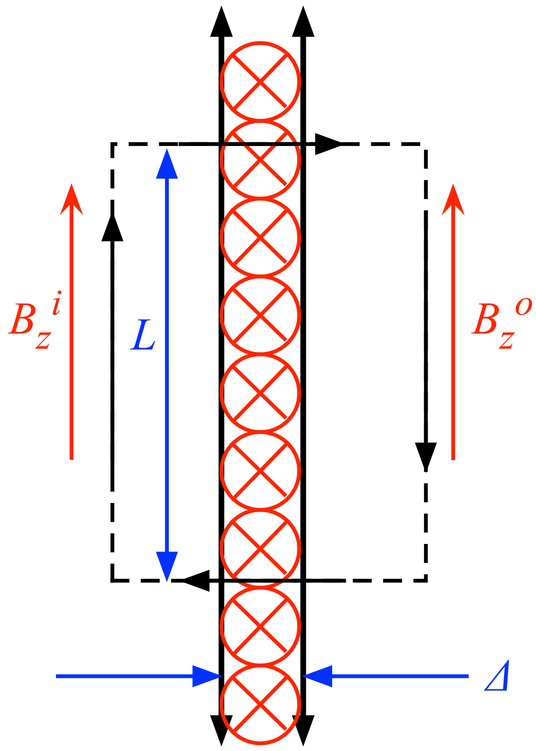

FIG. 5. Ampere's Law is used to integrate around the loop (shown as dashed rectangle) of length L.

B_z^o is the drive field and I will solve Eq. 19 for a drive field that is a step function

$$B_z^o = u(t) B_z \tag{20}$$

where B_z is a constant magnetic field and $u(t)$ is the Heaviside operator. With this drive and the initial condition $B_z^i = 0$ at $t = 0$, the first order differential equation is easily solved using Laplace transforms. The solution is

$$B_z^i(t) = B_z \left(1 - e^{-t/\tau}\right) \tag{21}$$

Thus, the $1/e$ time is τ which I will call the diffusion time. (Note: I get the same result as Haus *et al* [4], example 10.3.1.)

III. COMPARING EDDY CURRENT EFFECTS BETWEEN THE DISK AND THE TUBE

The effect of the eddy current on the heating and magnetic field reduction between the disk and the tube will be considered here.

A. Heating effects

Now, I want to compare the heating between the disk and the tube. This is easy to do if I compare the maximum instantaneous power between the two geometries and also assume that the maximum values of $\dot{B}_z$ used in Eq. 9 and $\dot{B}_z^i$ used in Eq. 17 are equal, then I can write down the ratio

$$\left(\frac{P_{\text{tube}}}{P_{\text{disk}}}\right)_{\text{max}} = \frac{4L\Delta}{hR} \tag{22}$$

Therefore, for the losses on the tube be much smaller than the disk, I have the following inequality

$$L\Delta \ll hR/4 \tag{23}$$

And I can choose the thickness of the shell $\Delta = h/4$, and the inequality becomes

$$L \ll R \tag{24}$$

Unfortunately, in our geometry, L is about the same size as R and so the heating in the tube can be just as much as the disk!

B. Magnetic field reduction

For the disk, the eddy current reduces the size of the drive field B_z by $-\mu_0\sigma\dot{B}_z hR/4$ at the centre of the disk. Therefore a thinner disk or lower conductivity metal will reduce this effect.

For the tube, there is a $1/e$ time given by $\tau = \mu_0\sigma R\Delta/2$. This can reparametrized as a bandwidth and I can write down the 3 dB point from the $1/e$ time and it is

$$f_{3\text{dB}} = \frac{1}{2\pi\tau} = \frac{1}{\pi\mu_0\sigma R\Delta} \tag{25}$$

I can put in the numbers for copper, (see Eq. 2) and put $\Delta = 10^{-3}$ m and $R = 0.2$ m to get $f_{3\mathrm{dB}} = 21$ Hz. This is not good when compared to our magnetic field ramp frequency of 15 Hz.

IV. CONCLUSION

These calculations show that the eddy current on the walls of the tube are not negligible! In fact, it can be as large as that on the end wall disk if we don't put in cuts. The reduction of the B-field that arise from eddy current effects must also be mitigated by cuts.

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[1] H.E. Knoepfel. *Magnetic fields: A comprehensive theoretical treatise for practical use*, page 187. John Wiley and Sons, 200.

[2] M. Krakowski. Eddy-current losses in thin circular and rectangular plates. *Archiv für Elektrotechnik*, 64:307–311, 1982.

[3] J.P. Poltz. On eddy currents in thin plates. *Archiv für Elektrotechnik*, 66:225–229, 1983.

[4] H.A. Haus and J.R. Melcher. *Electromagnetic fields and energy*, chapter 10. Prentice-Hall, New Jersey, 1989.